

Possible Connection between Probability, Spacetime Geometry and Quantum Mechanics

Enrique Canessa*

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

Abstract

Following our discussion [Physica A **375** (2007) 123] to associate an analogous probabilistic description with spacetime geometry in the Schwarzschild metric from the macro- to the micro-domain, we argue that there is a possible connection among normalized probabilities \mathcal{P} , spacetime geometry (in the form of Schwarzschild radii r_s) and quantum mechanics (in the form of complex wave functions ψ), namely $\sqrt{\mathcal{P}_{\theta,\phi,t}^{(n)}} \approx R_s^{(n)}/r_s = |\psi_n^{(n)}(X^{(n)})|^2/|\psi_n(x)|^2$. We show how this association along different (n) -nested surfaces –representing curve space due to an inhomogeneous density of matter– preserves the postulates of quantum mechanics at different geometrical scales.

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* e-mail: canessae@ictp.it

I. OVERVIEW

In a recent paper (hereafter denoted as I) [1], we argued how the universe we live in (consisting of an inhomogeneous density of matter in the form of stars, molecules, atoms, *etc*, that curve space due to their gravitational fields) may be seen fulfilled with nested surfaces. In I we considered Schwarzschild's isotropic metric –which is the base for tests of general relativity and of the existence of black holes– in the vicinity of multiple massive objects. Within this picture, we demonstrated that the probabilistic description of stochastic processes of a general *birth-and-death* model can be associated to the Schwarzschild metric at different geometrical scales.

Inspired by this idealization (which in I it was considered a toy model since there were no output variables), our goal in this work is to argue that there is a possible connection between an analogous probability, spacetime curvature and quantum mechanics. Within this association, the postulates of quantum mechanics are preserved on scales ranging from the size of the observable universe to micro-world distances greater than the black hole horizon. We use same notation as in I and write down the algebra in details.

II. PROBABILITY AND SPACETIME GEOMETRY

Let us start with the standard isotropic form of Schwarzschild's metric [2]

$$ds^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \quad , \quad (1)$$

where $m \equiv GM/c^2$, M is the mass of the body producing the field, G is the Gravitational constant, c is the speed of light and $d\Omega^2 = d\theta^2 + \sin\theta \, d\phi^2$ is the element of solid angle. Infinitesimal radial distances in this metric, at fixed polar angles θ and ϕ and time t , then satisfy

$$dR \equiv \left(1 - \frac{2m}{r}\right)^{-1/2} dr \quad . \quad (2)$$

In I, we extended this equation to a system of nested curved surfaces S_m forming a spiral of n -interconnected curved surfaces along distance scales from the macro- to the micro-world. In other words we assumed the space to be curved at all geometric scales $dR^{(n)}$ due to the presence of scattered matter $M^{(i)} \neq 0$ ($i = 0, 1, \dots, n$) with $M^{(0)} \equiv m$. We then established

the master relation

$$\left(\frac{dr}{dR^{(n)}}\right)^2 \equiv \mathcal{P}_{\theta,\phi,t}^{(n)}(r) < 1 \quad . \quad (3)$$

This relates space geometry to an iterative (n -process) probability-like function. We proved rigorously in I that the function $\mathcal{P}^{(n)}$ can be considered as an analogous probability measure since it implies well-defined stationary states to exist and its sum satisfies the normalization condition $\sum_{i=0}^n \mathcal{P}^{(i)} = \sum_{i=0}^n \left(dr/dR^{(i)}\right)^2 \equiv 1$ over n different states.

The analogous probabilities \mathcal{P} , at fixed polar angles and time, are obtained recursively from the *birth-and-death*-like general recursive relation

$$\mathcal{P}^{(n+1)}(r) = \left(\frac{\lambda^{(n)}}{\mu^{(n+1)}}\right) \mathcal{P}^{(n)}(r) \rightleftharpoons \left(1 - \frac{R_s^{(n+1)}}{R^{(n)}}\right) \mathcal{P}^{(n)}(r) \quad , \quad (4)$$

such that

$$\begin{aligned} \mathcal{P}^{(n)}(r) &\rightleftharpoons \frac{\lambda^{(n-1)} \dots \lambda^{(1)} \lambda^{(0)}}{\mu^{(n)} \dots \mu^{(2)} \mu^{(1)}} \mathcal{P}^{(0)}(r) \quad , \\ \mathcal{P}^{(0)}(r) &\rightleftharpoons 1 - \frac{r_s}{r} \quad , \\ \lambda^{(n-1)} &\rightleftharpoons (R^{(n-1)} - R_s^{(n)})/r_s \\ \mu^{(n)} &\rightleftharpoons R^{(n-1)}/r_s \quad . \end{aligned} \quad (5)$$

In the above $R_s^{(n)} = 2GM^{(n)}/c^2$ are Schwarzschild radii –at which the metric of eq.(2) becomes singular for positive $n \geq 1$, $R_s^{(0)} \equiv r_s = 2m$, $R^{(0)} \equiv R$, and (λ, μ) are *birth-and-death* coefficients, respectively.

Using forward and backward two-point approximations for the derivative of the scattered matter $M^{(n+1)} - M^{(n)} \approx dM^{(n)}/dn$ and $(\mu^{(n+1)} - \mu^{(n)}) - (\lambda^{(n)} - \lambda^{(n-1)}) \approx (d\mu^{(n)}/dn) - (d\lambda^{(n)}/dn)$, in I we also suggested that

$$\frac{1}{M} \left(\frac{dM^{(n)}}{dn}\right) \approx \frac{1}{r_s} \left(\frac{dR_s^{(n)}}{dn}\right) \rightleftharpoons \frac{d\mu^{(n)}}{dn} - \frac{d\lambda^{(n)}}{dn} \approx \frac{1}{\mathcal{E}} \left(\frac{d\mathcal{E}^{(n)}}{dn}\right) \quad . \quad (6)$$

These relations imply that variations of an analogous system energy (and mass) coefficient can be related to: *i*) the difference between some 'annihilated' analogous processes of leaving state n and those being 'generated' in the state n , or *ii*) some small (stochastic) fluctuations in the Schwarzschild radii.

III. POSSIBLE CONNECTION

Let us extend the main ideas behind eq.(3) –*i.e.*, the concept of analogous probability as being consistent with the full filling of the curved space with nested surfaces– to a gen-

eral (non necessarily stochastic) 1D process. We keep r and R as independent variables at different geometrical scales correlated by the Schwarzschild metric. We also consider $P^{(0)}(r) = \text{constant}$, which is the case in eq.(5) for distances greater than the Schwarzschild radii r_s .

The coordinate distance (length) is here approximated to lie radially in the field of each n -spherical object of mass $M^{(n)}$ (see, *e.g.*, [2]), namely $dr \rightarrow \Delta r = x - x_0$. At each nested surface we approximate $dR^{(n)} \rightarrow \Delta R^{(n)} = X^{(n)} - X_0^{(n)}$. Our master equation therefore simplifies to

$$\frac{dr}{dR^{(n)}} \rightarrow \frac{\Delta r}{\Delta R^{(n)}} = \frac{x - x_0}{X^{(n)} - X_0^{(n)}} \equiv \sqrt{\mathcal{P}^{(n)}} < 1 \quad . \quad (7)$$

For Δ 's small enough, and choosing the origin at $x_0 = X_0^{(n)} \equiv 0$, we can estimate the ratio of (micro to macro) potential fields as

$$\frac{dr}{dR^{(n)}} \rightarrow \frac{\Delta r}{\Delta R^{(n)}} \approx \left(\frac{x}{-GMM^{(n)}} \right) \cdot \left(\frac{-GMM^{(n)}}{X^{(n)}} \right) \equiv \frac{U^{(n)}}{U} = \sqrt{\mathcal{P}^{(n)}} \quad , \quad (8)$$

with $U \neq 0$. The system masses are independent of x as deduced from eq.(6). That is,

$$\frac{M^{(n)}}{M} \approx \frac{R_s^{(n)}}{r_s} \rightleftharpoons \mu^{(n)} - \lambda^{(n)} \approx \frac{\mathcal{E}^{(n)}}{\mathcal{E}} \quad . \quad (9)$$

Any constant factor in the integration is taken to be zero.

For further progress, it is worth to check our associations against quantum physics measurements, at least up to a submolecular magnitude of the order of the Bohr radius $x \rightarrow a_0 = \hbar^2/me^2 \sim 0.529\text{\AA}$ [3, 4]. Considering circular orbits in the Bohr atom model, the quantized angular-momentum is assumed to be $L \equiv n\hbar$ (with $L^2 = me^2r$), and the allowed orbital radii are given by $r_n = n^2a_0$. The corresponding energy is $E_n = -e^2/2a_0n^2$ with $n = 1, 2, \dots$. These relations can be rewritten as

$$\left(\frac{a_0}{r_n} \right)^2 = \left(\frac{r_1}{r_n} \right)^2 = \frac{1}{n^4} \quad ; \quad \frac{E_n}{E_1} = \frac{1}{n^2} \quad . \quad (10)$$

In conjunction with our eqs.(7) and (9), for $n > 1$ we can readily identify

$$\left(\frac{x}{X^{(n)}} \right)^2 = \mathcal{P}^{(n)} \propto \frac{1}{n^4} < 1 \quad ; \quad \frac{M^{(n)}}{M} \approx \frac{\mathcal{E}^{(n)}}{\mathcal{E}} \rightarrow \frac{1}{n^2} \propto \sqrt{\mathcal{P}^{(n)}} \quad , \quad (11)$$

which leads the association $M^{(n)}/M \rightleftharpoons x/X^{(n)}$. This, in turn, implies $M^{(n)}e^2X^{(n)} \rightleftharpoons Me^2x$ which means that the angular momentum is conserved. An estimate of L can be obtained by considering circular orbits of radius $X^{(n)}$ around $x \rightarrow a_0$, *i.e.*, $L^2 = Me^2X^{(n)} =$

$(\hbar^2/a_0)(x/\sqrt{\mathcal{P}^{(n)}})$. In conjunction with eq.(11), the angular-momentum quantization becomes $L = \hbar/(\mathcal{P}^{(n)})^{1/4} \rightarrow n\hbar$. This is just an illustrative example since Schwarzschild's metric is valid for non-rotating bodies only. The results of the Bohr empirical model given in eq.(11) are our motivation though to depict the general Hamiltonian treatment of quantum mechanics.

Let us derive the kinetics energy operator $\hat{K} \equiv -(\hbar^2/2M)(d^2/dx^2)$ in 1D, by applying again eq.(3) and the second derivative of x discussed in the Appendix. Using also eq.(11) for the scattered mass, it follows then that

$$\hat{K} \equiv \frac{-\hbar^2}{2M} \left(\frac{d^2}{dx^2} \right) = \frac{-\hbar^2 \sqrt{\mathcal{P}^{(n)}}}{2M^{(n)}} \cdot \frac{1}{\mathcal{P}^{(n)}} \left(\frac{d^2}{dX^{(n)2}} \right) = \frac{-\hbar^2}{\sqrt{\mathcal{P}^{(n)}} 2M^{(n)}} \left(\frac{d^2}{dX^{(n)2}} \right) \equiv \frac{\hat{K}^{(n)}}{\sqrt{\mathcal{P}^{(n)}}} . \quad (12)$$

This association, together with eq.(8), leads to

$$\hat{\mathcal{E}}^{(n)} \equiv \hat{K}^{(n)} + U^{(n)} \rightarrow \sqrt{\mathcal{P}^{(n)}} (\hat{K} + U) = \sqrt{\mathcal{P}^{(n)}} \hat{\mathcal{E}} . \quad (13)$$

A result that extends eq.(11) for the total system energy \mathcal{E} to an energy operator at different iterations. Therefore from the definition of the time-independent quantum mechanics Hamiltonian $\hat{K}\psi_n + U\psi_n \equiv \hat{H}\psi_n = \hat{\mathcal{E}}_n\psi_n$, with $\psi(x)$ a quantized system wave function and $\hat{\mathcal{E}}_n$ representing energy states, we see that

$$\hat{\mathcal{E}}_n^{(n)}\psi_n^{(n)} = \hat{H}^{(n)}\psi_n^{(n)} \rightarrow \hat{H}\psi_n = \hat{\mathcal{E}}_n\psi_n , \quad (14)$$

provided the wave function in each nested surface satisfies, $\psi_n^{(n)}(X^{(n)}) \rightleftharpoons \Lambda^{(n)}\psi_n(x)$. (In the right hand side of eq.(13) the factor $\sqrt{\mathcal{P}^{(n)}}$ cancels out and Λ is a non zero function to be defined next). ψ is allowed to be complex.

Using the probability density postulate of quantum mechanics and the relation between dx and $dX^{(n)}$ of eq.(7), we immediately get

$$d\Pi_n = |\psi_n(x)|^2 dx \rightleftharpoons \left| \frac{\psi_n^{(n)}(X^{(n)})}{\Lambda^{(n)}} \right|^2 \sqrt{\mathcal{P}^{(n)}} dX^{(n)} = \frac{\sqrt{\mathcal{P}^{(n)}}}{\Lambda^{(n)2}} d\Pi_n^{(n)} . \quad (15)$$

Therefore in order to preserve both the general Hamiltonian form of eq.(14) and the probability density $\Pi_n = \Pi_n^{(n)}$ -at the *n-essima* nested surface and at the same quantum state n , we must to have

$$\Lambda^{(n)} = (\mathcal{P}^{(n)})^{1/4} \rightleftharpoons \psi_n^{(n)}(X^{(n)})/\psi_n(x) . \quad (16)$$

This, in turn, leads to the correct mathematical requirement of normalization for the wave functions

$$\int |\psi_n(x)|^2 dx \Leftrightarrow \int \left| \frac{\psi_n^{(n)}(X^{(n)})}{(\mathcal{P}^{(n)})^{1/4}} \right|^2 \sqrt{\mathcal{P}^{(n)}} dX^{(n)} = \int |\psi_n^{(n)}(X^{(n)})|^2 dX^{(n)} \equiv 1 \quad . \quad (17)$$

For a system in a state described by the normalized wave function above, the expectation value of an observable corresponding to A is also conserved independent of the chosen nested surface. In fact,

$$\begin{aligned} \langle A \rangle &\equiv \int \psi_n^*(x) \hat{A} \psi_n(x) dx \Leftrightarrow \int \left(\frac{\psi_n^{(n)*}(X^{(n)})}{(\mathcal{P}^{(n)})^{1/4}} \right) \hat{A} \left(\frac{\psi_n^{(n)}(X^{(n)})}{(\mathcal{P}^{(n)})^{1/4}} \right) \sqrt{\mathcal{P}^{(n)}} dX^{(n)} \\ &= \int \psi_n^{(n)*}(X^{(n)}) \hat{A} \psi_n^{(n)}(X^{(n)}) dX^{(n)} \quad . \end{aligned} \quad (18)$$

Finally, it can be seen that the property of orthogonality (a general result for quantum mechanical eigenfunctions [3, 4]) is also preserved

$$\begin{aligned} \int \psi_m(x) \psi_n(x) dx &= \delta_{m,n} \Leftrightarrow \int \left(\frac{\psi_m^{(n)}(X^{(n)})}{(\mathcal{P}^{(n)})^{1/4}} \right) \left(\frac{\psi_n^{(n)}(X^{(n)})}{(\mathcal{P}^{(n)})^{1/4}} \right) \sqrt{\mathcal{P}^{(n)}} dX^{(n)} \\ &= \int \psi_m^{(n)}(X^{(n)}) \psi_n^{(n)}(X^{(n)}) dX^{(n)} \quad . \end{aligned} \quad (19)$$

IV. REMARKS

In order to verify that the present connection between an analogous probability, spacetime geometry and quantum mechanics makes sense, let us consider the results of the Schrödinger equation for a particle of mass M in a box of length L . This simplest nontrivial model, having potential energy $U = U^{(n)} \equiv 0$, illustrates many of the fundamental concepts of quantum mechanics [3, 4]. The model predictions are: energy levels $E_n = (\hbar^2/8ML^2)n^2$ and normalized eigenfunctions $\psi_n(x) = (2/L)^{1/2} \sin(n\pi x/L)$, with $n = 1, 2, \dots$

By applying our associations such that $L/L^{(n)} \rightarrow x/X^{(n)} = \sqrt{\mathcal{P}^{(n)}} \Leftrightarrow M^{(n)}/M$, it follows that for the particle in a box

$$\psi_n(x) = \left(\frac{2}{\sqrt{\mathcal{P}^{(n)}} L^{(n)}} \right)^{1/2} \sin \left(\frac{n\pi \sqrt{\mathcal{P}^{(n)}} X^{(n)}}{\sqrt{\mathcal{P}^{(n)}} L^{(n)}} \right) = \frac{\psi_n(X^{(n)})}{(\mathcal{P}^{(n)})^{1/4}} \quad , \quad (20)$$

and

$$E_n = \left(\frac{\sqrt{\mathcal{P}^{(n)}} \hbar^2}{8M^{(n)} (\sqrt{\mathcal{P}^{(n)}} L^{(n)})^2} \right) n^2 = \frac{E_n^{(n)}}{\sqrt{\mathcal{P}^{(n)}}} \quad . \quad (21)$$

Both of these relations are compatible with eqs.(16) and (13), respectively. These results emphasize our suggestion to perceive spacetime in terms of surfaces interconnected due to the presence of dispersed mass even at atomic levels. From eq.(11), we have in our notation that $M^{(0)} > M^{(1)} > \dots > M^{(n)}$ (hence, $x < X^{(n)}$) and $dr < dR < dR^{(1)} < \dots < dR^{(n)}$. This simply means that the system of nested curved surfaces form a spiral (*c.f.*, Fig. 1 in I).

The normalized analogous probabilities $\mathcal{P}^{(n)}$ in eq.(3) are valid for fixed time and polar angles. According to eqs.(9) and (11) –or eq.(13), we have that $\mathcal{P}^{(n)} = (\mu^{(n)} - \lambda^{(n)})^2$ when considering general stochastic *birth-and-death* processes. Furthermore, we have considered $P^{(0)}$ to be constant in order to evaluate derivatives –*e.g.*, those in the Appendix. This is the case in eq.(5) for $r > r_s$. Here the label n relates quantum states and the upper symbol (n) accounts for the *n-essima* interconnected surface. The meaning of $P^{(n)}$ is different from the meaning of $\Pi^{(n)}$. The latter relates the probability density postulate of quantum mechanics in eq.(15). Throughout different geometrical scales, we have explicitly shown that our associations lead to preserve the Hamiltonian form (*c.f.*, eq.(14)) and postulates of quantum mechanics (*c.f.*, eqs.(17)-(19))

To summarize, we derived along different (n) -nested surfaces the following novel connection among normalized analogous probabilities, spacetime geometry (in the form of Schwarzschild radii) and quantum mechanics (in the form of complex wave functions)

$$\sqrt{\mathcal{P}_{\theta,\phi,t}^{(n)}} \approx \frac{R_s^{(n)}}{r_s} = \frac{|\psi_n^{(n)}(X^{(n)})|^2}{|\psi_n(x)|^2} . \quad (22)$$

We believe this association could be useful to analyze quantum mechanics processes above the event horizon, or quantum systems that are macroscopic both in their spatial dimensions and in the number of particles involved [5], each of which causes curvature of the spacetime around it.

At the apparent horizon of a black hole that has the Schwarzschild metric, or inside and around the horizon (where the role of time and space coordinates is interchanged), quantum fluctuations are involved in the process of Hawking radiation [6]. It is interesting to note that Hawking predicted that a black hole radiates thermally like a hot coal, with a temperature T inversely proportional to its mass. For a black hole of solar mass ($M = 1.99 \times 10^{30} Kg$), this implies $T \sim 10^{-6} K$ –which is negligible at the present age of the universe [7]. But for a black hole of a mountain of mass $M^{(n)} \sim 10^{12} Kg$, $T^{(n)} \sim 10^{12} K$ which is hot enough to emit photons, electrons and positrons. Hence we estimate the same order of magnitude

(i.e., 10^{-18}) for $R_s^{(n)}/R_s = M^{(n)}/M \sim 10^{12}/10^{30} \Rightarrow (1/T^{(n)})/(1/T) \sim 10^{-6}/10^{12}$. This could be another compelling reason for interest in the present work.

Appendix

To obtain the kinetics energy operator \hat{K} given in eq.(12) we use eq.(3) and consider the second derivative

$$\begin{aligned} \frac{d^2}{dx^2} &= \frac{d}{dx} \left(\frac{d}{dx} \right) = \frac{d}{\left(\frac{dx}{dX^{(n)}} \right) dX^{(n)}} \left(\frac{d}{\left(\frac{dx}{dX^{(n)}} \right) dX^{(n)}} \right) \\ &\approx \frac{1}{\sqrt{\mathcal{P}^{(n)}}} \frac{d}{dX^{(n)}} \left(\frac{1}{\sqrt{\mathcal{P}^{(n)}}} \frac{d}{dX^{(n)}} \right) = \frac{1}{\mathcal{P}^{(n)}} \left(\frac{d^2}{dX^{(n)2}} \right) . \end{aligned} \quad (23)$$

The last term is obtained from the fact that we are deriving with respect to the independent variable $X^{(n)}$ and that the analogous probability \mathcal{P} -function depends on $X^{(n-1)}$ (as can be deduced from eq.(5)). A similar relation follows by applying eq.(3) directly to $d^2/dx^2 = d^2/(\mathcal{P}^{(n)}dX^{(n)2})$.

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- [1] E. Canessa, *Physica A* **375** (2007) 123.
 - [2] J. Foster and J.D. Nightingale in "*A Short Course in General Relativity*" (Springer-Verlag, NY, USA, 1994).
 - [3] S.M. Blinder, in "*Introduction to Quantum Mechanics*" (Elsevier Acad. Press, MA, USA, 2004).
 - [4] C. Cohen-Tannoudji, B. Diu and F. Laloë in "*Quantum Mechanics*", Vol. I and II (John Wiley & Sons., NY, USA, 1977).
 - [5] K. Kong Wan in "*From Micro to Macro Quantum Systems*" (World Scientific Pub. Co. Inc., Singapore, 2006).
 - [6] M. Kenmoku, H. Kubotani, E. Takasugi and Y. Yamazaki, *Phys. Rev. D* **57** (1998) 4925.
 - [7] B.J. Carr and S. B. Giddings, *Sci. Am.*, April 2005.